

# New integral representations of $n$ th order convex functions

Teresa Rajba

Department of Mathematics and Computer Science, University of Bielsko-Biala, ul. Willowa 2, 43-309 Bielsko-Biala, Poland

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## Abstract

In this paper we give an integral representation of an  $n$ -convex function  $f$  in general case without additional assumptions on function  $f$ . We prove that any  $n$ -convex function can be represented as a sum of two  $(n + 1)$ -times monotone functions and a polynomial of degree at most  $n$ . We obtain a decomposition of  $n$ -Wright-convex functions which generalizes and complements results of Maksa and Pales [12]. We define and study relative  $n$ -convexity of  $n$ -convex functions. We introduce a measure of  $n$ -convexity of  $f$ . We give a characterization of relative  $n$ -convexity in terms of this measure, as well as in terms of  $n$ th order distributional derivatives and Radon-Nikodym derivatives. We define, study and give a characterization of strong  $n$ -convexity of an  $n$ -convex function  $f$  in terms of its derivative  $f^{(n+1)}(x)$  (which exists a.e.) without additional assumptions on differentiability of  $f$ . We prove that for any two  $n$ -convex functions  $f$  and  $g$ , such that  $f$  is  $n$ -convex with respect to  $g$ , the function  $g$  is the support for the function  $f$  in the sense introduced by Wasowicz [27], up to polynomial of degree at most  $n$ .

**Keywords:** higher-order convexity, higher-order Wright-convexity, strong convexity, relative convexity, multiple monotone function, support theorems

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## 1. Introduction

The notion of  $n$ th order convexity (or  $n$ -convexity) was defined in terms of divided differences by Popoviciu [20] (cf. also [22], [11]), however, we will not state it here. Instead we list some definitions of  $n$ th order convexity which are equivalent to the Popoviciu's definition.

**Proposition 1.1.** *A function  $f(x)$  is  $n$ -convex on  $(a, b)$  ( $n \geq 1$ ) if and only if its derivative  $f^{(n-1)}(x)$  exists and is convex on  $(a, b)$  (with the convention  $f^{(0)}(x) = f(x)$ ).*

This fact first was proved by Hopf [9, p. 24] and by Popoviciu [20, p. 38] (see also [11], [22]). Many results on  $n$ -convex functions one can found, among others, in [10], [1], [6], [11], [22], [14], [17], [26], [27], [28], [5].

Recall that convex functions satisfy various smoothness properties. A convex function defined on  $(a, b)$  is continuous and has both right and left derivatives  $f'_R(x)$  and  $f'_L(x)$  at each point of  $(a, b)$ . In addition both these derivatives are non-decreasing and satisfy inequality  $f'_L(x) \leq f'_R(x)$  for all  $x \in (a, b)$  (see [22], [11]). Thus we have

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Email address: trajba@ath.bielsko.pl (Teresa Rajba)

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**Proposition 1.2.** *A function  $f(x): (a, b) \rightarrow \mathbb{R}$  is  $n$ th order convex ( $n \geq 1$ ) if and only if its right derivative  $f_R^{(n)}(x)$  (or left derivative  $f_L^{(n)}(x)$ ) exists and is non-decreasing on  $(a, b)$ .*

If  $f(x)$  is sufficiently smooth on  $[a, b]$ , then from Taylor's Theorem we have

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!} \int_a^b (x-t)_+^n f^{(n)}(t) dt,$$

where  $(x-t)_+^{n-1} = \max\{(x-t)^{n-1}, 0\}$ .

Now assume  $f(x)$  is  $n$ th order convex on  $(a, b)$  ( $n \geq 1$ ). Then the left and right derivatives  $f_L^{(n)}(x)$  and  $f_R^{(n)}(x)$  exist on  $(a, b)$ . In addition, both these functions are non-decreasing. With each such  $f$  we associate the measure  $\mu$  defined on  $(a, b)$  by

$$\mu([x, y]) = f_R^{(n)}(y) - f_L^{(n)}(x),$$

for  $a < x \leq y < b$ . This is a non-negative Borel measure on  $(a, b)$ . If  $f_R^{(n)}(a)$  is finite then  $\mu$  can be extended to a bounded (finite) measure on whole  $[a, c]$ , for all  $c < b$ . In this case  $f(x)$  has the representation

$$f(x) = \sum_{k=0}^n \frac{f_R^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!} \int_a^b (x-t)_+^n d\mu(t),$$

for  $x \in (a, b)$ . If we cannot extend  $\mu$  to the endpoint  $a$ , then we will have this representation only on closed subintervals of  $(a, b)$ . The converse also holds. These results can be found in Popoviciu [20] (see also Karlin and Studden [10], Bullen [3], Brown [2], Granata [6], Pinkus and Wulbert [17]). In other words, the above integral representation is valid for all  $x \in (a, b)$  if  $\mu$  is of bounded variation on  $(a, b)$ , otherwise we have this representation only on closed subinterval of  $(a, b)$ .

In this paper we give an analogue of the integral representation above in general case. The representation we obtain deals with measures  $\mu$  with not necessarily bounded variations. Our characterization is constructive. We give explicit formulas for  $n$ -spectral measures corresponding to an  $n$ -convex function in this representation (see Section 2).

The strength of the representation developed in Section 2 is exploited in the rest of the paper. It is used to further study of  $n$ -convexity, and to obtain complete characterizations of strong  $n$ -convexity,  $n$ -Wright-convexity, and relative  $n$ -convexity of functions, among other. Finally, the representation is employed to examine support-type properties of  $n$ -convex functions.

In Section 3 we prove that an  $n$ -convex function can be represented as a sum of two  $(n+1)$ -times monotone functions and a polynomial of degree at most  $n$ . This result generalizes the well-known theorem on representation of a convex function as a sum of non-increasing and non-decreasing functions, and a polynomial of degree at most 1 (see Roberts and Varberg [22]). Using our decomposition we obtain the decomposition of  $n$ -Wright-convex functions, which generalizes and complements results of Maksa and Páles [12].

In Section 4 we define and study relative  $n$ -convexity of  $n$ -convex functions. Relative  $n$ -convexity induces the partial ordering in the set of  $n$ -convex functions. We define a measure of  $n$ -convexity of an  $n$ -convex function  $f$  using  $n$ -spectral measures in our representation. We give a characterization of relative  $n$ -convexity in terms of the measure of  $n$ -convexity, as well as in terms of  $n$ th order distributional derivatives, and in terms of Radon-Nikodym derivatives. Using the Lebesgue decomposition of  $n$ -spectral measures corresponding to an  $n$ -convex function  $f$ ,

we consider the corresponding decomposition of the function  $f$ . This decomposition is applied to derive some useful characterizations of the relative  $n$ -convexity.

We define and study the notion of strong  $n$ -convexity that generalizes the strong convexity. It is well known that the strong convexity of a function  $f$  can be characterized in terms of its second derivative  $f''(x)$  for twice differentiable  $f$ . We give a characterization of the strong  $n$ -convexity of an  $n$ -convex function  $f$  in terms of only derivative  $f^{(n+1)}(x)$  (which exists almost everywhere with respect to Lebesgue measure), without any additional assumptions on differentiability of  $f$ .

In Section 5 we obtain a generalization of Wasowicz [27] results. We prove, that for any two  $n$ -convex functions  $f$  and  $g$ , such that  $f$  is  $n$ -convex with respect to  $g$ , the function  $g$  is the support for the function  $f$  in the sense introduced by Wasowicz [27], up to a polynomial of degree at most  $n$ .

## 2. Integral representation

In this chapter we give an integral representation of an  $n$ -convex function  $f$  without additional assumptions on  $f$ . We derive explicit formulas for  $n$ -spectral measures corresponding to  $f$  that can be applied to measures of not necessary bounded variation on  $(a, b)$ .

By  $\lambda$  we denote the Lebesgue measure. Let  $\Pi_n$  be the family of all polynomials of degree at most  $n$ . Let  $f: (a, b) \rightarrow \mathbb{R}$  be an  $n$ th order convex function on the interval  $(a, b)$ , where  $-\infty \leq a < b \leq \infty$ ,  $n = 1, 2, \dots$ . Then  $f_R^{(n)}(x)$  is non-decreasing and right-continuous on  $(a, b)$ . Henceforth  $f^{(n)}(x)$  will be used to denote  $f_R^{(n)}(x)$ . A function  $f^{(n)}(x)$  must satisfy one of the following three conditions:

- A. There exist  $x_1, x_2 \in (a, b)$  such that  $f^{(n)}(x_1) < 0$  and  $f^{(n)}(x_2) > 0$ ,
- B.  $f^{(n)}(x) \geq 0$  for all  $x \in (a, b)$ ,
- C.  $f^{(n)}(x) \leq 0$  for all  $x \in (a, b)$ .

**Theorem 2.1.** *For  $n \geq 1$  each  $n$ th order convex function  $f: (a, b) \rightarrow \mathbb{R}$  satisfying the property A admits the representation of the form*

$$f(x) = \int_{(a, \xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} dg_{(n)-}(u) + \int_{[\xi, b)} \frac{(x-u)_+^n}{n!} dg_{(n)+}(u) + Q(x), \quad (2.1)$$

where  $\xi \in (a, b)$ ,  $g_{(n)-} \leq 0$  is a non-decreasing right-continuous function on  $(a, b)$ ,  $g_{(n)+}$  is a non-decreasing left-continuous function on  $(a, b)$  such that  $g_{(n)+}(\xi) = g_{(n)-}(\xi) = 0$ , and  $Q \in \Pi_{n-1}$ . Moreover, the functions  $g_{(n)-}$ ,  $g_{(n)+}$  and  $Q$  are determined uniquely,  $g_{(n)+} = f_+^{(n)}$  a.e.,  $g_{(n)-} = f_-^{(n)}$  a.e.

**Notation 2.2.** *The quantities*

$$\Psi_{(n)-}(x) = \Psi_{(n)-}(x; a, \xi, dg_{(n)-}(u)) = \int_{(a, \xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} dg_{(n)-}(u), \quad (2.2)$$

$$\Psi_{(n)+}(x) = \Psi_{(n)+}(x; \xi, b, dg_{(n)+}(u)) = \int_{[\xi, b)} \frac{(x-u)_+^n}{n!} dg_{(n)+}(u) \quad (2.3)$$

appear frequently and hence from now we will be using the above notation.

**Remark 2.3.** A straightforward calculation shows that

$$\begin{aligned}\frac{d^n}{dx^n}\Psi_{(n)-}(x; a, \xi, dg_{(n)-}(u)) &= \int_{(a, \xi]} [-\chi_{(-\infty, 0)}(x-u)] dg_{(n)-}(u) \\ &= g_{(n)-}(x) \text{ a.e. } (x \in (a, \xi)),\end{aligned}\quad (2.4)$$

$$\begin{aligned}\frac{d^n}{dx^n}\Psi_{(n)+}(x; \xi, b, g_{(n)+}(u)) &= \int_{[\xi, b]} \chi_{(0, \infty)}(x-u) dg_{(n)+}(u) \\ &= g_{(n)+}(x) \text{ a.e. } (x \in (\xi, b)).\end{aligned}\quad (2.5)$$

**PROOF (PROOF OF THEOREM 2.1).** Let  $f$  be an  $n$ th order convex function satisfying the property A. Then there exists  $\xi \in (a, b)$  such that  $f^{(n)}(\xi+) \geq 0$  and  $f^{(n)}(\xi-) \leq 0$ . Let  $g_{(n)-}(x)$  and  $g_{(n)+}(x)$  be right-continuous and left-continuous functions, respectively, and such that

$$g_{(n)-}(x) = \min\{0, f^{(n)}(x)\}, g_{(n)+}(x) = \max\{0, f^{(n)}(x)\} \text{ a.e.} \quad (2.6)$$

Then  $g_{(n)-}(\xi) = g_{(n)+}(\xi) = 0$ . From (2.4), (2.5) and (2.6) we obtain that the functions  $f(x)$  and  $\Psi_{(n)-}(x) + \Psi_{(n)+}(x)$  differ on  $(a, b)$  by a polynomial of degree at most  $n-1$ . Thus (2.1) is satisfied. Conversely, assume  $f$  is of the form (2.1). By Remark 2.3,  $f^{(n)}_-(x) = g_{(n)-}(x)$  and  $f^{(n)}_+(x) = g_{(n)+}(x)$  a.e. Thus  $f^{(n)}(x)$  is non-decreasing and right-continuous on  $(a, b)$ . This implies that  $f(x)$  is  $n$ th order convex on  $(a, b)$ . The proof is completed.

**Remark 2.4.** Note that since  $[-(x-u)]_+^n = 0$  for  $u < x$ , and  $(x-u)_+^n = 0$  for  $u > x$ , the integral (2.2) is over  $[x, \xi]$  and the integral (2.3) is over  $[\xi, x]$ . Since  $dg_{(n)-}(u)$  and  $dg_{(n)+}(u)$  are of bounded variations on  $(x, \xi)$  and  $(\xi, x)$ , respectively, the integrals in (2.2) and (2.3) are well defined.

**Remark 2.5.** If  $g_{(n)-}(b-) = 0$ , then in (2.4) we set  $\xi = b$ . Similarly if  $g_{(n)+}(a+) = 0$ , then we put  $\xi = a$  in (2.5).

**Theorem 2.6.** For  $n \geq 1$  each  $n$ -convex function  $f: (a, b) \rightarrow \mathbb{R}$  satisfying the property B admits the representation

$$f(x) = \int_a^b \frac{(x-u)_+^n}{n!} dg_{(n)}(u) + Q(x), \quad (2.7)$$

where  $Q(x) = c_n x^n/n! + \dots + c_0$ ,  $c_n \geq 0$ , and  $g_{(n)}(x)$  is a non-negative non-decreasing left-continuous function on  $(a, b)$  satisfying  $g_{(n)}(a+) = 0$ . Moreover,  $Q(x)$  and  $g_{(n)}(x)$  are uniquely determined,  $c_n = f^{(n)}(a+)$ ,  $g_{(n)}(x) = f^{(n)}(x) - c_n$  a.e., and  $Q(x) = f(x) - \psi_{(n)+}(x; a, b, dg_{(n)}(u))$ .

**PROOF.** Assume  $f$  is  $n$ -convex function such that  $f^{(n)}(x) \geq 0$  ( $x \in (a, b)$ ). Taking into account that  $f^{(n)}(x)$  is non-negative and non-decreasing on  $(a, b)$ ,  $f^{(n)}(a+) = c_n$  exists and is finite. Let  $g_{(n)}(x)$  be a left-continuous function such that  $g_{(n)}(x) = f^{(n)}(x) - c_n$  a.e. ( $x \in (a, b)$ ). Then  $g_{(n)}(x)$  is non-negative, non-decreasing, and satisfies  $g_{(n)}(a+) = 0$ . In view of Remark 2.5, by (2.5) with  $a$  in place of  $\xi$  and  $g_{(n)}(x)$  in place of  $g_{(n)+}(x)$ , we have

$$\frac{d^n}{dx^n}\psi_{(n)+}(x; a, b, dg_{(n)}(u)) = g_{(n)}(x) \text{ a.e. } (x \in (a, b)).$$

Consequently

$$\frac{d^n}{dx^n}\psi_{(n)+}(x; a, b, dg_{(n)}(u)) = f^{(n)}(x) - c_n \text{ a.e. } (x \in (a, b)).$$

Thus the functions  $\psi_{(n)+}(x; a, b, dg_{(n)}(u))$  and  $f(x) - c_n x^n/n!$  differ on  $(a, b)$  by a polynomial of degree at most  $(n-1)$ . The theorem is proved.

**Theorem 2.7.** For  $n \geq 1$  each  $n$ th order convex function  $f: (a, b) \rightarrow \mathbb{R}$  satisfying the property C admits the representation of the form

$$f(x) = \int_a^b (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} dg_{(n)}(u) + Q(x), \quad (2.8)$$

where  $Q(x) = c_n x^n / n! + \dots + c_0$ ,  $c_n \leq 0$ , and  $g_{(n)}(x)$  is a non-positive non-decreasing right-continuous function on  $(a, b)$  such that  $g_{(n)}(b-) = c_n$ . Moreover,  $g_{(n)}$  and  $Q$  are uniquely determined,  $c_n = f^{(n)}(b-)$ ,  $g_{(n)}(x) = f^{(n)}(x) - c_n$  a.e., and  $Q(x) = f(x) - \psi_{(n)-}(x; a, b, dg_{(n)}(u))$ .

PROOF. The proof is similar to the proof of Theorem 2.6 and hence it is omitted.

**Remark 2.8.** The representations (2.1), (2.7) and (2.8) can be rewritten in equivalent forms using the following two measures associated with the distribution functions  $g_{(n)-}(x)$  and  $g_{(n)+}(x)$ , defined as

$$\begin{aligned} \mu_{(n)-}(du) &= dg_{(n)-}(u), \\ \mu_{(n)+}(du) &= dg_{(n)+}(u). \end{aligned}$$

We will call  $\mu_{(n)-}$  and  $\mu_{(n)+}$  the  $n$ -spectral measures of an  $n$ -convex function  $f$ .

The following theorem summarizes Theorems 2.1, 2.6 and 2.7.

**Theorem 2.9.**

a) For  $n \geq 1$  each  $n$ -convex function  $f: (a, b) \rightarrow \mathbb{R}$  admits the representation

$$f(x) = \int_{(a, \xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)-}(du) + \int_{[\xi, b)} \frac{(x-u)_+^n}{n!} \mu_{(n)+}(du) + Q(x), \quad (2.9)$$

where  $\xi \in [a, b]$ . Moreover, if  $f^{(n)}(x)$  satisfies the condition B (or C), then  $\xi = a$ ,  $\mu_{(n)-} = 0$  and  $\mu_{(n)+}(du) = d(f^{(n)}(u) - f^{(n)}(a+))$  (or  $\xi = b$ ,  $\mu_{(n)+} = 0$  and  $\mu_{(n)-}(du) = d(f^{(n)}(u) - f^{(n)}(b-))$ ), and if  $f^{(n)}(x)$  satisfies the condition A, then  $\xi \in (a, b)$ ,  $f^{(n)}(\xi-) \leq 0$ ,  $f^{(n)}(\xi+) \geq 0$ ,  $\mu_{(n)-}(du) = df_-^{(n)}(u)$ ,  $\mu_{(n)+}(du) = df_+^{(n)}(u)$  and  $Q(x) \in \Pi_n$ .

b) If  $f^{(n)}(a+) = \alpha$  exists and is finite, then  $f(x)$  can be rewritten in the form

$$f(x) = \int_a^b \frac{(x-u)_+^n}{n!} \mu_{(n)a+}(du) + Q_a(x),$$

where  $\mu_{(n)a+}(du) = d(f^{(n)}(u) - \alpha)_+$ ,  $Q_a(x) \in \Pi_n$ .

c) If  $f^{(n)}(b-) = \beta$  exists and is finite, then  $f(x)$  can be rewritten in the form

$$f(x) = \int_a^b (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)b-}(du) + Q_b(x),$$

where  $\mu_{(n)b-}(du) = d(f^{(n)}(u) - \beta)_-$ ,  $Q_b(x) \in \Pi_n$ .

Denoting

$$\begin{aligned} \psi_f(x) &= \psi_{(n)f}(x) = \\ &= \int_{(a, \xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)-}(du) + \int_{[\xi, b)} \frac{(x-u)_+^n}{n!} \mu_{(n)+}(du), \end{aligned}$$

(2.9) can be rewritten in the form

$$f(x) = \psi_f(x) + Q(x).$$

Note, that every function  $f(x)$  can be trivially written as  $f(x) = f(x) - cx^n/n! + cx^n/n! (c \in \mathbb{R})$ . Thus  $f(x)$  can be also written in the form

$$f(x) = \psi_{f-cx^n/n!}(x) + Q_c(x), \quad (2.10)$$

where  $Q_c(x) \in \Pi_n$ .

Another representation is given in the following theorem. This representation is important in applications of the theory to study relative  $n$ -convexity.

**Theorem 2.10.** *Let  $f: (a, b) \rightarrow \mathbb{R}$  be an  $n$ -convex function. For every  $\xi \in (a, b)$  the function  $f(x)$  has the representation*

$$f(x) = \int_{(a, \xi]} (-1)^{n+1} \frac{[-(x-u)]_+^n}{n!} \mu_{(n)\xi-}(du) + \int_{[\xi, b)} \frac{(x-u)_+^n}{n!} \mu_{(n)\xi+}(du) + Q_\xi(x), \quad (2.11)$$

where

$$\begin{aligned} \mu_{(n)\xi-}(du) &= d[f^{(n)}(u) - f^{(n)}(\xi+)]_-, \\ \mu_{(n)\xi+}(du) &= d[f^{(n)}(u) - f^{(n)}(\xi+)]_+, \\ Q_\xi &\in \Pi_n. \end{aligned} \quad (2.12)$$

Moreover, we have

$$\mu_{(n)\xi-} + \mu_{(n)\xi+} = \mu_{(n)-} + \mu_{(n)+}, \quad (2.13)$$

where  $\mu_{(n)-}$  and  $\mu_{(n)+}$  are the  $n$ -spectral measures corresponding to  $f$ .

**PROOF.** Let  $a < \xi < b$ . Put  $c = f^{(n)}(\xi+)$  and denote  $g_c(x) = f(x) - cx^n/n!$ . Then  $g_c^{(n)}(\xi-) \leq 0$  and  $g_c^{(n)}(\xi+) \geq 0$ , and consequently the function  $g_c^{(n)}(x)$  satisfies the condition A. By Theorem 2.9 with  $g_c(x)$  in place of  $f(x)$ , and taking into account (2.10), we obtain the representation (2.11) with the measures  $\mu_{(n)\xi-}$  and  $\mu_{(n)\xi+}$  satisfying (2.12). It implies that  $f^{(n)}(x) - f^{(n)}(\xi+)$  is the distribution function corresponding to the sum of measures  $\mu_{(n)\xi-} + \mu_{(n)\xi+}$ . By Theorem 2.9, the distribution function corresponding to the sum of  $n$ -spectral measures  $\mu_{(n)-} + \mu_{(n)+}$  equals  $f^{(n)}(x)$  up to a constant. Thus the distribution functions of measures  $\mu_{(n)\xi-} + \mu_{(n)\xi+}$  and  $\mu_{(n)-} + \mu_{(n)+}$  differ on  $(a, b)$  by a constant. Consequently these measures coincide, so (2.13) is proved.

### 3. $n$ -convexity and multiple monotonicity

From Theorem 2.9 on the representation of an  $n$ -convex function  $f$  we obtain that  $f$  can be represented by the sum of two  $(n+1)$ -times monotone functions and a polynomial of degree at most  $n$ . Applying this we obtain a theorem on decomposition of an  $n$ -Wright-convex function, which complements and generalizes results of Maksa and Páles [12].

By the standard definition (cf. Williamson [29]) a function  $f: (a, b) \rightarrow \mathbb{R}$  is called  $n$ -times monotone non-increasing ( $n \geq 2$ ) if  $(-1)^k f^{(k)}(x)$  is non-negative, non-increasing, and convex for  $x \in (a, b)$  and  $k = 0, 1, \dots, n-2$ . When  $n = 1$ ,  $f(x)$  is simply non-negative and non-increasing.

The well-known representation for  $n$ -times monotone non-increasing functions on  $(0, \infty)$  states that

$$f(x) = \int_0^\infty (1 - ux)_+^{n-1} d\beta(u) \quad (x > 0), \quad (3.1)$$

with  $\beta(u)$  being non-decreasing (see Williamson [29]).

A function  $f$  is called  *$n$ -times monotone non-decreasing* (briefly  *$n$ -times monotone*) ( $n \geq 2$ ) if  $f^{(k)}(x)$  is non-negative, non-decreasing, and convex for  $x \in (a, b)$  and  $k = 0, 1, 2, \dots, n-2$ . When  $n = 1$ ,  $f(x)$  is simply non-negative and non-decreasing. From (3.1) we derive the following representations of functions  $f: (a, b) \rightarrow \mathbb{R}$ , which are  $(n+1)$ -times monotone non-increasing and  $(n+1)$ -times monotone non-decreasing on  $(a, b)$ , respectively:

$$f(x) = \int_a^b \frac{(x-u)_+^n}{n!} d\beta(u), \quad (3.2)$$

$$f(x) = \int_a^b \frac{[-(x-u)]_+^n}{n!} d\beta(u), \quad (3.3)$$

where  $\beta(u)$  is non-decreasing.

We point out that the representation (3.2) has a short proof. Also, without loss of generality we may assume that  $a = -\infty$  and  $b = \infty$ .

**Theorem 3.1.** *Let  $n \geq 1$ . A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  satisfying the condition  $f(-\infty) = 0$  is  $n$ -times monotone non-decreasing if and only if it admits the representation*

$$f(x) = \int_{-\infty}^\infty \frac{(x-u)_+^{n-1}}{(n-1)!} d\beta(u), \quad (3.4)$$

where  $\beta(u)$  is non-decreasing and  $\beta(-\infty) = 0$ . Moreover,  $\beta(u)$  is unique at its points of continuity and  $\beta(u) = f^{(n-1)}(u)$  a.e.

PROOF. The sufficiency is evident by differentiating (3.4), for

$$f^{(k)}(x) = \int_{-\infty}^\infty \frac{(x-u)_+^{n-1-k}}{(n-1-k)!} d\beta(u),$$

( $k = 0, 1, \dots, n-1$ ) is evidently non-negative and non-decreasing.

To see the necessity let us consider  $\beta(u) = f^{(n-1)}(u)$  (with the convention  $f^{(0)}(x) = f(x)$ ). Then (3.4) can be rewritten as

$$f(x) = \int_{-\infty}^x \frac{(x-u)^{n-1}}{(n-1)!} df^{(n-1)}(u). \quad (3.5)$$

We prove it by induction.

Let  $n = 1$ . Since  $f(x)$  is non-decreasing and  $f(-\infty) = 0$ , we have

$$f(x) = \int_{-\infty}^x df(u). \quad (3.6)$$

This proves (3.5) for  $n = 1$ .

Assume that (3.5) holds for some  $n \geq 1$ , i.e.  $n$ -times monotone non-decreasing functions  $f(x)$  such that  $f(-\infty) = 0$  are of the form (3.5). Let  $f(x)$  (such that  $f(-\infty) = 0$ ) be an  $(n+1)$ -times monotone non-decreasing, i.e.  $f(x)$  is  $n$ -times monotone non-decreasing and  $f^{(n)}(x)$  is non-decreasing. It is not difficult to show that  $f^{(n)}(-\infty) = 0$ . By (3.5) and (3.6), with  $f^{(n)}(x)$  in place of  $f(x)$ , we have

$$\begin{aligned} f(x) &= \int_{-\infty}^x \frac{(x-u)^{n-1}}{(n-1)!} df^{(n-1)}(u) \\ &= \int_{-\infty}^x \frac{(x-u)^{n-1}}{(n-1)!} f^{(n)}(u) du \\ &= \int_{-\infty}^x \frac{(x-u)^{n-1}}{(n-1)!} \int_{-\infty}^u df^{(n)}(v) du \\ &= \int_{-\infty}^x \int_{-\infty}^x \frac{(x-u)^{n-1}}{(n-1)!} du df^{(n)}(v) \\ &= \int_{-\infty}^x \frac{(x-v)^n}{n!} df^{(n)}(v). \end{aligned}$$

This proves (3.5), with  $n$  replaced by  $n+1$ , so the induction is complete.

Let  $\mathcal{M}_{n+}((a, b))$  ( $\mathcal{M}_{n-}((a, b))$ ) be the class of all  $n$ -times monotone non-decreasing (non-increasing) functions on  $(a, b)$ . Taking into account (3.2) and (3.3), by Theorem (2.9) we obtain the following decomposition.

**Theorem 3.2.** *Let  $n \geq 1$  and  $f: (a, b) \rightarrow \mathbb{R}$ . Then  $f$  is  $n$ th order convex if and only if  $f$  is of the form*

$$f(x) = M_1(x) + M_2(x) + Q(x),$$

where  $(-1)^{n+1}M_1(x) \in \mathcal{M}_{(n+1)-}((a, \xi))$ ,  $M_2(x) \in \mathcal{M}_{(n+1)+}((\xi, b))$ , with  $a \leq \xi \leq b$ ,  $Q(x) = c_n x^n/n! + \dots + c_0 \in \Pi_n$ . Moreover, if  $M_1 \equiv 0$  and  $M_2 \not\equiv 0$  then  $c_n \geq 0$ , if  $M_2 \equiv 0$  and  $M_1 \not\equiv 0$  then  $c_n \leq 0$ , and if  $M_1 \not\equiv 0$  and  $M_2 \not\equiv 0$  then  $c_n = 0$  and  $M_1(\xi-) = M_2(\xi+)$ .

**Remark 3.3.** Note that if  $g(x) \in \mathcal{M}_{(n+1)+}((a, b))$ , then  $\varphi(x) = g(-x) \in \mathcal{M}_{(n+1)-}((-b, -a))$ . Thus  $M_2(x) = g(x)$  is  $n$ th order convex on  $(a, b)$  and  $M_1(x) = (-1)^{n+1}\varphi(x)$  is  $n$ th order convex on  $(-b, -a)$ .

The following proposition gives a characterization of  $n$ -times monotone non-decreasing functions in terms of difference operators (see McNeil [13]).

**Proposition 3.4.** *Let  $n \geq 1$  and  $f: (a, b) \rightarrow \mathbb{R}$ . Then the following statements are equivalent.*

- (i)  $f$  is  $n$ -times monotone on  $(a, b)$ .
- (ii)  $f$  is non-negative and for any  $k = 1, \dots, n$ , any  $x \in (a, b)$ , and any  $h_i > 0$ ,  $i = 1, \dots, k$  such that  $x + h_1 + \dots + h_k \in (a, b)$  the function  $f$  satisfies

$$\Delta_{h_k} \dots \Delta_{h_1} f(x) \geq 0, \quad (3.7)$$

where  $\Delta_{h_k} \dots \Delta_{h_1}$  denote sequential applications of the first-order difference operator  $\Delta_h$  given by  $\Delta_h f(x) = f(x+h) - f(x)$  whenever  $x, x+h \in (a, b)$ .



(iii)  $f$  is non-negative and satisfies, for any  $k = 1, \dots, n$ , any  $x \in (a, b)$ , and any  $h > 0$  such that  $x + kh \in (a, b)$

$$(\Delta_h)^k f(x) \geq 0, \quad (3.8)$$

where  $(\Delta_h)^k$  denote the  $k$ -monotone sequential iterations of the operator  $\Delta_h$ .

Note that in Gilanyi and Páles [5] functions satisfying (3.7) with  $k = n + 1$  are called *Wright-convex of order  $n$*  (or simply  *$n$ -Wright-convex*). As it is extensively discussed in [11] ( $n \geq 1$ ), the functions satisfying (3.8) with  $k = n + 1$  are called *Jensen-convex of order  $n$* . It is well-known that, under the assumption of continuity, Jensen convexity of order  $n$  and  $n$ th order convexity are equivalent. In the study of inequalities (3.7) and (3.8), functions that satisfy (3.7) and (3.8) with equality play a crucial role. For  $n \in \mathbb{N}$ , a function  $P: \mathbb{R} \rightarrow \mathbb{R}$  is called a *polynomial function of degree at most  $n$*  if it satisfies the Fréchet equation, i.e. if

$$(\Delta_h)^{n+1} P(x) = 0 \quad (h, x \in \mathbb{R}).$$

Polynomials are exactly the continuous polynomial functions, however, in terms of Hamel bases, one can construct non-continuous polynomial functions (see [11]). Maksa and Páles [12] proved that any  $n$ -Wright-convex function can be represented as the sum of a continuous  $n$ -convex function and a polynomial function

**Proposition 3.5.** *Let  $n \geq 1$  and  $f: (a, b) \rightarrow \mathbb{R}$ . Then  $f$  is an  $n$ -Wright-convex function if and only if  $f$  is of the form*

$$f(x) = C(x) + P(x) \quad (x \in (a, b)), \quad (3.9)$$

where  $C: (a, b) \rightarrow \mathbb{R}$  is a continuous  $n$ -convex function and  $P: \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial function of degree at most  $n$  with  $P(\mathbb{Q}) = \{0\}$ . Furthermore, under the assumption  $P(\mathbb{Q}) = 0$ , the decomposition (3.9) is unique.

From Theorem 3.2 and Proposition 3.5 we obtain the following decomposition of  $n$ -Wright-convex functions.

**Theorem 3.6.** *Let  $n \geq 1$  and  $f: (a, b) \rightarrow \mathbb{R}$ . Then  $f$  is an  $n$ -Wright-convex if and only if  $f$  is of the form*

$$f(x) = M_1(x) + M_2(x) + Q(x) + P(x),$$

where  $(-1)^{n+1}M_1(x)$  is an  $(n + 1)$ -times monotone non-increasing on  $(a, \xi)$ ,  $M_2(x)$  is an  $(n + 1)$ -times monotone non-decreasing on  $(\xi, b)$ ,  $a \leq \xi \leq b$ ,  $M_1(x) + M_2(x)$  is continuous on  $(a, b)$ ,  $Q(x)$  is a polynomial of degree at most  $n$  (as in Theorem 3.2) and  $P(x)$  is a polynomial function of degree at most  $n$ .

#### 4. Relative $n$ -convexity. Strong $n$ -convexity.

Let  $g: (a, b) \rightarrow \mathbb{R}$  be an  $n$ -convex function. We say that a function  $f: (a, b) \rightarrow \mathbb{R}$  is  *$n$ -convex with respect to  $g$*  if  $f - g$  is  $n$ -convex, and denote it by  $f \geq_n g$ .

Various other generalizations of convexity via related convexity properties have been proposed. The relative  $n$ -convexity defined above is a generalization of the relative convexity (for  $n = 1$ ) studied in Karlin and Studden [10] (cf. [4], [7], [15], [16]).

**Remark 4.1.** If  $f$  is  $n$ -convex with respect to  $g$ , then both  $f - g$  and  $g$  are  $n$ -convex. Writting  $f = g + (f - g)$ , we obtain that  $f$  necessarily must be  $n$ -convex.

Functions  $f: (a, b) \rightarrow \mathbb{R}$  and  $g: (a, b) \rightarrow \mathbb{R}$  that are decreasing and increasing on the same intervals will be called *isotonic*, and we say that they are members of the same isotonic class.

**Theorem 4.2.** *Let  $f, g: (a, b) \rightarrow \mathbb{R}$  be  $n$ -convex functions with  $n$ -spectral measures  $\mu_{(n)-}$ ,  $\mu_{(n)+}$  and  $\nu_{(n)-}$ ,  $\nu_{(n)+}$ , respectively. Then  $f$  is  $n$ -convex with respect to  $g$  if and only if*

$$\mu_{(n)} \geq \nu_{(n)}, \quad (4.1)$$

where

$$\begin{aligned} \mu_{(n)} &= \mu_{(n)-} + \mu_{(n)+}, \\ \nu_{(n)} &= \nu_{(n)-} + \nu_{(n)+}. \end{aligned}$$

PROOF. Let  $f$  and  $g$  satisfy the assumptions of the theorem. Fix  $a < \xi < b$ . By Theorem 2.10,  $f(x)$  and  $g(x)$  can be written in the form (2.11) with the measures  $\mu_{(n)\xi-}$ ,  $\mu_{(n)\xi+}$ ,  $\nu_{(n)\xi-}$  and  $\nu_{(n)\xi+}$ , and the polynomials  $Q_\xi(x)$  and  $R_\xi(x)$ , respectively. In other words, the functions  $f(x) - Q_\xi(x)$  and  $g(x) - R_\xi(x)$  are isotonic. Moreover, by (2.13), we have

$$\mu_{(n)-} + \mu_{(n)+} = \mu_{(n)\xi-} + \mu_{(n)\xi+}, \quad \nu_{(n)-} + \nu_{(n)+} = \nu_{(n)\xi-} + \nu_{(n)\xi+}. \quad (4.2)$$

Therefore  $f(x) - g(x)$  is of the form (2.11), with the measures  $\mu_{(n)\xi-} - \nu_{(n)\xi-}$  and  $\mu_{(n)\xi+} - \nu_{(n)\xi+}$  in the place of  $\mu_{(n)\xi-}$  and  $\mu_{(n)\xi+}$ , respectively, and  $Q_\xi(x) - R_\xi(x)$  in the place of  $Q_\xi(x)$ . By Theorem 2.10,  $f - g$  is  $n$ -convex if and only if

$$\mu_{(n)\xi-} \geq \nu_{(n)\xi-}, \quad \mu_{(n)\xi+} \geq \nu_{(n)\xi+}.$$

From (4.2) we conclude (4.1). The theorem is proved.

Theorem 4.2 suggests that we can define a *measure of  $n$ th order convexity* of an  $n$ -convex function by the operator

$$K: f \rightarrow \mu_{(n)}^f = \mu_{(n)} = \mu_{(n)-} + \mu_{(n)+}.$$

In the sequel we will call  $\mu_{(n)}^f$  the *measure of  $n$ -convexity of  $f$* , or shortly the  *$n$ -convexity measure*. From Theorem 4.2 we have

**Theorem 4.3.**

$$f \geq_n g \text{ if and only if } \mu_{(n)}^f \geq \mu_{(n)}^g.$$

We shall say that functions  $f, g: (a, b) \rightarrow \mathbb{R}$  are of *modulo  $\Pi_n$* , or that they are members of the same *modulo  $\Pi_n$  class*, if they differ by a polynomial  $Q \in \Pi_n$ . The relation modulo  $\Pi_n$  is an equivalence relation and hence it defines equivalence classes. For  $n$ -convex  $f$  and  $g: (a, b) \rightarrow \mathbb{R}$  that are members of the same modulo  $\Pi_n$  class we therefore have that  $f^{(n)}(x)$  and  $g^{(n)}(x)$  differ on  $(a, b)$  by a constant. Consequently, by Theorem 2.10, we have the following theorem

**Theorem 4.4.**

$$f = g \pmod{\Pi_n} \quad \text{if and only if} \quad \mu_{(n)}^f = \mu_{(n)}^g.$$

We now show that this relation induces a partial ordering.

**Theorem 4.5.** *The relative  $n$ -convexity relation induces a partial ordering on modulo  $\Pi_n$  equivalence classes of  $n$ -convex functions.*

PROOF. We will show that the relation is reflective, antisymmetric, and transitive.

**Reflectivity.** For all  $f$  we have  $f - f \equiv 0 \in \Pi_n$ . Thus  $f \succeq_n f$ .

**Antisymmetry.** Suppose  $f \succeq_n g$  and  $g \succeq_n f$ . Then  $f - g$  and  $g - f$  are  $n$ -convex. Thus, both functions  $(f - g)^{(n)}(x)$  and  $[-(f - g)]^{(n)}(x)$  are non-decreasing. Consequently,  $(f - g)^{(n)}(x) = 0$  ( $x \in (a, b)$ ). This implies that  $f - g \in \Pi_n$ , that is  $f = g \pmod{\Pi_n}$ .

**Transitivity.** Suppose  $f \succeq_n g$  and  $g \succeq_n h$ . Then both  $f - g$  and  $g - h$  are  $n$ -convex. Writing  $f - h$  in the form  $f - h = (f - g) + (g - h)$ , we obtain that  $f - h$  is  $n$ -convex as the sum of the  $n$ -convex functions. The theorem is proved.

As a simple example of the use of Theorem 2.10 we prove the following theorem. We denote by  $d\mu/d\nu = \varphi$  the Radon-Nikodym derivative of a measure  $\mu$  with respect to a measure  $\nu$  (see [23]).

**Theorem 4.6.** *Let  $f, g: (a, b) \rightarrow \mathbb{R}$  be  $n$ -convex. Then*

a) *there exists an  $n$ -convex function  $f_{\max}$  such that*

$$f_{\max} \succeq_n f, \quad f_{\max} \succeq_n g,$$

*and for every  $n$ -convex function  $h$*

$$(h \succeq_n f \text{ and } h \succeq_n g) \Rightarrow h \succeq_n f_{\max},$$

b) *there exists an  $n$ -convex function  $f_{\min}$  such that*

$$f \succeq_n f_{\min}, \quad g \succeq_n f_{\min},$$

*and for every  $n$ -convex function  $h$*

$$(f \succeq_n h \text{ and } g \succeq_n h) \Rightarrow f_{\min} \succeq_n h,$$

c) *if  $f \succeq_n g$  and  $f \not\equiv g \pmod{\Pi_n}$ , then there exists an  $n$ -convex function  $w$  such that  $f \not\equiv w \pmod{\Pi_n}$ ,  $g \not\equiv w \pmod{\Pi_n}$  and*

$$f \succeq_n w \succeq_n g.$$

PROOF. Let  $f$  and  $g$  be  $n$ -convex. By Theorem 2.10 we may assume that  $f$  and  $g$  admit representations given by (2.11) with the same  $\xi \in (a, b)$ , the measures  $\mu_{(n)\xi-}^f, \mu_{(n)\xi+}^f, \mu_{(n)\xi-}^g, \mu_{(n)\xi+}^g$ , and with the polynomials  $Q_\xi^f$  and  $Q_\xi^g$ , respectively. Consider the Radon-Nikodym derivatives

$$\varphi_1 = d\mu_{(n)\xi-}^f / d(\mu_{(n)\xi-}^f + \mu_{(n)\xi-}^g),$$

$$\psi_1 = d\mu_{(n)\xi-}^g / d(\mu_{(n)\xi-}^f + \mu_{(n)\xi-}^g),$$

$$\varphi_2 = d\mu_{(n)\xi+}^f / d(\mu_{(n)\xi+}^f + \mu_{(n)\xi+}^g),$$

$$\psi_2 = d\mu_{(n)\xi+}^g / d(\mu_{(n)\xi+}^f + \mu_{(n)\xi+}^g).$$

It is not difficult to see that it suffices to take the functions  $f_{\max}$  and  $f_{\min}$  of the form (2.11) with the measures

$$\mu_{(n)\xi-}^{\max} = \max(\varphi_1, \psi_1)(\mu_{(n)\xi-}^f + \mu_{(n)\xi-}^g),$$

$$\begin{aligned}\mu_{(n)\xi+}^{max} &= \max(\varphi_2, \psi_2)(\mu_{(n)\xi+}^f + \mu_{(n)\xi+}^g), \\ \mu_{(n)\xi-}^{min} &= \max(\varphi_1, \psi_1)(\mu_{(n)\xi-}^f + \mu_{(n)\xi-}^g), \\ \mu_{(n)\xi+}^{min} &= \max(\varphi_2, \psi_2)(\mu_{(n)\xi+}^f + \mu_{(n)\xi+}^g),\end{aligned}$$

to prove parts a) and b).

To prove c) assume  $f \geq_n g$  and  $f \neq g \pmod{\Pi_n}$ . Then  $f-g$  is  $n$ -convex and  $f-g \neq 0 \pmod{\Pi_n}$ . Thus it suffices to take  $w = g + \frac{1}{2}(f - g)$ . The theorem is proved.

As usual we denote distributional derivatives by  $f'$  (see [24], [25]), pointwise derivatives by  $f'(x)$ ,  $n$ th order distributional derivatives by  $f^{(n)}$ , and  $n$ th order pointwise derivatives by  $f^{(n)}(x)$ . Theorem (4.2) suggests that we can use distributional derivatives and the Radon-Nikodym derivatives to derive simple criteria for the relative  $n$ -convexity  $f \geq_n g$ .

**Theorem 4.7.** *Let  $f, g: (a, b) \rightarrow \mathbb{R}$  be  $n$ -convex functions with the  $n$ -convexity measures  $\mu_{(n)}^f$  and  $\mu_{(n)}^g$ , respectively. Then the following conditions are equivalent:*

- a)  $f \geq_n g$ ,
- b)  $\mu_{(n)}^f \geq \mu_{(n)}^g$ ,
- c)  $f^{(n+1)} \geq g^{(n+1)}$ ,
- d)  $d\mu_{(n)}^g/d\mu_{(n)}^f \leq 1$ .

Via Lebesgue's decomposition theorem and the decomposition of a singular measure, every  $\sigma$ -finite measure  $\mu$  can be decomposed into a sum of an absolutely continuous measure (with respect to the Lebesgue measure), a singular continuous measure, and a discrete measure, i.e.

$$\mu = \mu_{cont} + \mu_{sing} + \mu_{pp},$$

where  $\mu_{cont}$  is the absolutely continuous part,  $\mu_{sing}$  is the singular continuous part and  $\mu_{pp}$  is the pure point part (a discrete measure) (see Royden [23]). These three measures are uniquely determined.

**Remark 4.8.** The following decomposition yields an analogous decomposition of an  $n$ -convex function. Namely, any  $n$ -convex function  $f$  with the  $n$ -spectral measures  $\mu_{(n)-}$  and  $\mu_{(n)+}$  can be represented as a sum

$$f = f_{cont} + f_{sing} + f_{pp}, \quad (4.3)$$

where  $f_{cont}$ ,  $f_{sing}$  and  $f_{pp}$  correspond to the absolutely continuous parts the singular continuous parts, and the pure point parts of the  $n$ -spectral measures  $\mu_{(n)-}$  and  $\mu_{(n)+}$ , respectively ( $\mu_{(n)-} = \mu_{(n)-cont} + \mu_{(n)-sing} + \mu_{(n)-pp}$ ,  $\mu_{(n)+} = \mu_{(n)+cont} + \mu_{(n)+sing} + \mu_{(n)+pp}$ ). Note, that  $f_{cont}$ ,  $f_{sing}$  and  $f_{pp}$  are  $n$ -convex. Moreover, they are unique, up to a polynomial of degree at most  $n$ .

It is not difficult to prove the following lemma.

**Lemma 4.9.** *Let  $\mu$  and  $\nu$  be two  $\sigma$ -finite measures having the following decompositions into a sum of an absolutely continuous measure, a singular continuous measure and a discrete measure*

$$\mu = \mu_{cont} + \mu_{sing} + \mu_{pp} \quad \text{and} \quad \nu = \nu_{cont} + \nu_{sing} + \nu_{pp}.$$

*Then  $\mu \geq \nu$  if and only if  $\mu_{cont} \geq \nu_{cont}$ ,  $\mu_{sing} \geq \nu_{sing}$  and  $\mu_{pp} \geq \nu_{pp}$ .*

Taking into account the decomposition (4.3) of an  $n$ -convex function, by Lemma 4.9 we immediately obtain the following three theorems useful in studying relative  $n$ -convexity.

**Theorem 4.10.** *Let  $f$  and  $g: (a, b) \rightarrow \mathbb{R}$  be  $n$ -convex functions having the decompositions  $f = f_{cont} + f_{sing} + f_{pp}$ ,  $g = g_{cont} + g_{sing} + g_{pp}$  (see (4.3)). Then  $f \succeq_n g$  if and only if  $f_{cont} \succeq_n g_{cont}$  and  $f_{sing} \succeq_n g_{sing}$  and  $f_{pp} \succeq_n g_{pp}$ .*

**Theorem 4.11.**

$$f_{cont} \succeq_n g_{cont} \quad \text{iff} \quad f_{cont}^{(n+1)}(x) \geq g_{cont}^{(n+1)}(x)$$

**Theorem 4.12.**

$$f_{pp} \succeq_n g_{pp} \quad \text{iff} \quad f_{pp}^{(n+1)} \geq g_{pp}^{(n+1)},$$

where  $f_{pp}^{(n+1)} = \sum_k a_k \delta_{x_k}$ ,  $g_{pp}^{(n+1)} = \sum_k b_k \delta_{y_k}$ .

The notion of convexity can be extended not only to the case when the order of convexity is of higher-dimension, but also in several other ways. One of the most important generalizations is the notion of strong convexity. A function  $f: (a, b) \rightarrow \mathbb{R}$  is called *strongly convex with modulus*  $c > 0$  if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all  $x, y \in (a, b)$  and  $t \in [0, 1]$ . Strongly convex functions were introduced by Polyak [19]. Some properties of them can be found, among other, in [22], [8], [18]. Not attempting to be complete, we just recall here two results concerning strong convexity which play crucial roles in further investigations (see [22]). The first one characterizes strong convexity in terms of convexity, while the second one characterizes twice differentiable strongly convex function in terms of its second derivative  $f''(x)$ .

**Proposition 4.13.** *A function  $f: (a, b) \rightarrow \mathbb{R}$  is strongly convex with modulus  $c > 0$  if and only if the function  $f(x) - cx^2$  is convex.*

**Proposition 4.14.** *Assume that  $f: (a, b) \rightarrow \mathbb{R}$  is twice differentiable and  $c > 0$ . Then  $f$  is strongly convex with modulus  $c$  if and only if  $f''(x) \geq 2c$  ( $x \in (a, b)$ ).*

As a generalization of strong convexity with modulus  $c$ , we define strong  $n$ -convexity with modulus  $c$ . We say that a function  $f$  is *strongly  $n$ -convex with modulus  $c$*  ( $n \geq 1$ ,  $c > 0$ ) if  $f$  is  $n$ -convex with respect to the function  $g(x) = cx^{(n+1)}/(n+1)!$ . By Proposition 4.13 the strong convexity with modulus  $2c$  (cf Roberts and Varberg [22]) coincides with our strong 1-convexity with modulus  $c$ . Writing  $f(x) = (f(x) - cx^{n+1}/(n+1)!) + cx^{n+1}/(n+1)!$ , we obtain that if  $f$  is strongly  $n$ -convex with modulus  $c > 0$ , then  $f$  is  $n$ -convex.

The following theorem gives a characterization of a strongly  $n$ -convex function  $f$  with modulus  $c$  without additional assumptions on differentiability of  $f$ . This generalizes Proposition 4.14.

**Theorem 4.15.** *Let  $f: (a, b) \rightarrow \mathbb{R}$  be an  $n$ -convex function and  $c > 0$ . Then  $f$  is strongly  $n$ -convex with modulus  $c$  if and only if*

$$f^{(n+1)}(x) \geq c \text{ for } x \in (a, b) \quad \lambda \text{ a.e.}$$

PROOF. Note that the function  $g(x) = cx^{n+1}/(n+1)!$  is  $n$ -convex with the  $n$ -convexity measure  $\mu_{(n)}^g(dx) = dg^{(n)}(x) = cdx$ , where  $g^{(n)}(x) = cx$ . Writting  $g(x)$  in the form (4.3), we have  $g = g_{cont}$ ,  $g_{sing} = 0$  and  $g_{pp} = 0$ . As  $f$  is  $n$ -convex, we look at its integral representation given by (2.7), with the  $n$ -convexity measure  $\mu_{(n)}^f(dx) = df^{(n)}(x)$ . Since  $f^{(n)}(x)$  is nondecreasing its derivative  $f^{(n+1)}(x)$  exists for  $x \in (a, b)$   $\lambda$  a.e. By Remark 4.8  $f(x)$  can be represented as the sum

$$f = f_{cont} + f_{sing} + f_{pp}.$$

From Theorem 4.9, and taking into account that  $g_{cont} = g$ ,  $g_{sing} = g_{pp} = 0$ , we obtain that

$$f \geq_n g \text{ iff } f_{cont} \geq_n g_{cont}.$$

By Theorem 4.10,  $f_{cont} \geq_n g_{cont}$  iff  $f_{cont}^{(n+1)}(x) \geq g_{cont}^{(n+1)}(x)$ . Since  $g_{cont}^{(n+1)}(x) = c$ , and  $f_{cont}^{(n+1)}(x) = f^{(n+1)}(x)$  for  $x \in (a, b)$   $\lambda$  a.e., the theorem is proved.

**Corollary 4.16.** *Let  $c > 0$ ,  $n \in \mathbb{N}$  and  $f: (a, b) \rightarrow \mathbb{R}$  be a function. Then  $f$  is strongly  $n$ -convex with modulus  $c$  if and only if  $f$  is of the form*

$$f(x) = f_{cont}(x) + R(x) \quad (x \in (a, b)),$$

where  $f_{cont}: (a, b) \rightarrow \mathbb{R}$  is an  $(n+1)$ -times differentiable strongly  $n$ -convex function with modulus  $c$ , and  $R: (a, b) \rightarrow \mathbb{R}$  is an  $n$ -convex function such that  $R^{(n+1)}(x) = 0$  for  $x \in (a, b)$   $\lambda$  a.e.

**Corollary 4.17.** *Let  $c > 0$ ,  $n \in \mathbb{N}$  and let  $f: (a, b) \rightarrow \mathbb{R}$  be an  $(n+1)$ -times differentiable function. Then  $f$  is strongly  $n$ -convex with modulus  $c$  if and only if*

$$f^{(n+1)}(x) \geq c, \quad x \in (a, b).$$

## 5. Interpolation of functions by $n$ -convex functions.

It is well-known that every convex function  $f: I \rightarrow \mathbb{R}$  admits an affine support at every interior point of  $I$  (i.e. for any  $x_0 \in \text{Int}I$  there exists an affine function  $a: I \rightarrow \mathbb{R}$  such that  $a(x_0) = f(x_0)$  and  $a \leq f$  on  $I$ ). Convex functions of higher orders (precisely of an odd orders) have similar property; they are supported by polynomials of degree no greater than the order of convexity.

The following important property of convex functions of higher order (cf. Kuczma [11], Popoviciu [20], Roberts and Varberg [22]) is well-known: a function  $f: I \rightarrow \mathbb{R}$  is  $n$ -convex ( $I \subset \mathbb{R}$  is an interval) if and only if for any  $x_1, \dots, x_{n+1} \in I$  with  $x_1 < \dots < x_{n+1}$  the graph of an interpolating polynomial  $p := P(x_1, \dots, x_{n+1}; f)$  passing through the points  $(x_i, f(x_i))$ ,  $i = 1, \dots, n+1$ , changes successively from one side of the graph of  $f$  to another (always  $p(x) \leq f(x)$  for  $x \in I$  such that  $x > x_{n+1}$  if such points exist). More precisely,  $(-1)^{n+1}(f(x) - p(x)) \geq 0$  for  $x > x_{n+1}$ ,  $x \in I$ ,  $(-1)^{n+1-i}(f(x) - p(x)) \geq 0$ ,  $x_i < x < x_{i+1}$ ,  $i = 1, \dots, n$ ,  $f(x) - p(x) \geq 0$ ,  $x > x_{n+1}$ ,  $x \in I$ . It is not difficult to observe that the  $n$ -convexity reduces to convexity in the usual sense if  $n = 1$ .

In the Wasowicz paper [27] certain attaching method is developed. The method is applied in Theorem 5.1 to obtain a general result, from which the mentioned above support theorem and some related properties of convex functions of higher order are derived.

**Theorem 5.1.** Let  $n \in \mathbb{N}$  and  $f: I \rightarrow \mathbb{R}$  be an  $n$ -convex function. Let us fix  $k \in \mathbb{N}$ ,  $k \leq n$ , and take  $x_1, \dots, x_k \in I$  such that  $x_1 < \dots < x_k$ . To each point  $x_j$  ( $j = 1, \dots, k$ ) assign the multiplicity  $l_j \in \mathbb{N}$  such that  $l_1 + \dots + l_k = n + 1$ . Additionally assume that if  $x_1 = \inf I$ , then  $l_1 = 1$ , and if  $x_k = \sup I$ , then  $l_k = 1$ . Denote  $I_0 = (-\infty, x_1)$ ,  $I_j = (x_j, x_{j+1})$ ,  $j = 1, \dots, k-1$ , and  $I_k = (x_k, \infty)$ . Under these assumptions there exists a polynomial  $p \in \Pi_n$  such that  $p(x_j) = f(x_j)$ ,  $j = 1, \dots, k$ , and such that  $(-1)^{n+1}(f(x) - p(x)) \geq 0$  for  $x \in I_0 \cap I$ ,  $(-1)^{n+1-(l_1+\dots+l_j)}(f(x) - p(x)) \geq 0$  for  $x \in I_j$ ,  $j = 1, \dots, k-1$ ,  $f(x) - p(x) \geq 0$  for  $x \in I_k \cap I$ .

The numbers  $l_1, \dots, l_k$  can be interpreted as multiplicities of the points  $x_1, \dots, x_k$ , respectively. The polynomial  $p(x)$  in the above theorem will be called the *support* of  $(l_1, \dots, l_k)$ -type.

**Remark 5.2.** This fact is shown by Wasowicz [28] in a more general setting, i.e. for functions convex with respect to Chebyshev systems (for Chebyshev's polynomial system  $(1, x, \dots, x^n)$  such convexity reduces to  $n$ -convexity).

**Observation 5.3.** The polynomial  $p(x)$  described in Theorem 5.1 has following properties:

- (i)  $p(x) \leq f(x)$ ,  $x > x_k$ ,  $x \in I$ ,
- (ii) if  $l_j$  (i.e. the multiplicity of  $x_j$ ) is even, then the graph of  $p(x)$  passing through  $x_j$  remains on the same side of the graph of  $f$ , while it changes the side, if  $l_j$  is odd.

We apply Theorem 5.1 to obtain a general result, that for any two  $n$ -convex functions  $f$  and  $g$ , such that  $f$  is  $n$ -convex with respect to  $g$ , the function  $g$  is a support of  $(l_1, \dots, l_k)$ -type for function  $f$ , up to some polynomial  $p \in \Pi_n$ .

**Theorem 5.4.** Let  $n \in \mathbb{N}$  and let  $f$  and  $g: I \rightarrow \mathbb{R}$  be two  $n$ -convex functions such that  $f$  is  $n$ -convex with respect to  $g$ . Fix  $k \in \mathbb{N}$ ,  $k \leq n$ , and let  $x_1, \dots, x_k \in I$  be such that  $x_1 < \dots < x_k$ . Suppose that  $l_j$ ,  $I_j$  satisfy conditions of Theorem 5.1. Then there exists a polynomial  $p \in \Pi_n$ , such that

$$f(x_j) = g(x_j) + p(x_j), \quad j = 1, \dots, k, \quad (5.1)$$

and additionally

$$\begin{aligned} (-1)^{n+1}[f(x) - (g(x) + p(x))] &\geq 0 \quad \text{for } x \in I_0 \cap I \\ (-1)^{n+1-(l_1+\dots+l_j)}[f(x) - (g(x) + p(x))] &\geq 0 \quad \text{for } x \in I_j, j = 1, \dots, k-1, \\ f(x) - (g(x) + p(x)) &\geq 0 \quad \text{for } x \in I_k \cap I. \end{aligned} \quad (5.2)$$

The function  $g(x) + p(x)$  will be called the *support* of  $(l_1, \dots, l_k)$ -type for the function  $f$ .

**PROOF.** Since  $f(x)$  is  $n$ -convex with respect to  $g$ ,  $f(x) - g(x)$  is  $n$ -convex. Applying Theorem 5.1 with the function  $f(x) - g(x)$  in place of  $f(x)$ , we obtain that there exists a polynomial  $p \in \Pi_n$  such that

$$\begin{aligned} f(x_j) - g(x_j) &= p(x_j) \quad j = 1, \dots, k, \\ (-1)^{n+1}[(f(x) - g(x)) - p(x)] &\geq 0 \quad \text{for } x \in I_0 \cap I, \\ (-1)^{n+1-(l_1+\dots+l_j)}[(f(x) - g(x)) - p(x)] &\geq 0 \quad \text{for } x \in I_j, j = 1, \dots, k-1, \\ (f(x) - g(x)) - p(x) &\geq 0 \quad \text{for } x \in I_k \cap I. \end{aligned}$$

Thus (5.1) and (5.2) are satisfied. This completes the proof.

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